Student Name:______________________________

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Section F

F1.

Timmy is taking a round trip drive from Waterloo to Toronto. When traveling from Waterloo to Toronto, Timmy drives at an average speed of 80km/h. When driving back from Toronto to Waterloo, Timmy drives at an average speed of 120km/h. What was Timmy’s average speed throughout the entire trip?

Solution: This question makes use of the speed time formula (speed * time = distance) Let d be the distance between Waterloo and Toronto. Thus, Timmy will drive a total distance of 2d. He takes d/80 time to drive from Waterloo to Toronto. He will take d/120 time to drive from Toronto to Waterloo. In total he will take d/80 + d/120 = 5d/240 = d/48 time. Thus his average speed = 2d / (d/48) = 96km/h

Answer to F1: 96km/h

F2.

Let x and y be positive integer numbers such that x + y = 12. Find the maximum possible value of $x^2 + 4xy + y^2$.

Solution: Consider that $(x + y)^2 = x^2 + 2xy + y^2 = 144$. Therefore, to maximize the value of $x^2 + 4xy + y^2$, one must maximize the value of 2xy. From here, an application of AM-GM would show that $xy \leq (\frac{x+y}{2})^2 = 36$, and that the equality holds at x = 6, y = 6. AN alternative method would be noticing that $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$, and that since $(x+y)^2 = 144$ and $(x-y)^2$ is maximized when it = 0 or when x = y = 6. Thus, the maximum value of $x^2 + 4xy + y^2 = 144 + 2(36) = 216$.

Answer to F2: 216

F3

Consider that the product of the non-zero digits of 2024 are equal to 16, a perfect square. We call any number with this property “quirky”. How many positive quirky numbers there are less than 2024?

Student Name:______________________________

Please write your name on *every* page.

Consider how many non-zero/one digits are present in the number

0 non zero/one digits: There are 2 options for each digit. Accounting for the fact that 0 is not positive, there are thus $2^4 - 1 = 15$ quirky numbers where their digits product is 1

1 non-zero/one digits:

The digit can either be 4 or 9, one of the two one digit perfect squares. Thus for 9 and 4, there are $2^3 * 3 = 24$ possibilities for each of them. Thus, there are 2*24 = 48 total possibilities.

2 non-zero/one digits:

In addition to having 2 of the same digit, there is also the possibility of a number having the digits 4 and 9. Thus there are three scenarios:

If both digits are 2:
If one of the 2s is the first digit, then the second 2 has to be the 3rd or 4th digit and then the remaining digits are 0s or 1s. This gives $2 * 2^2 = 8$ possibilities. Otherwise, both of the two digits are in the last 3 digits, which has 3C2 = 3 possibilities. Since there are still other digits that can be 0 or 1, we have $2^2 * 3 = 12$ possibilities. Thus 20 in total.

If both digits are the same number:
Both of the digits are in the last 3 digits, which has 3C2 = 3 possibilities. Since there are still other digits that can be 0 or 1, we have $2^2 * 3 = 12$ possibilities. There are 7 possibilities of what the pair of digits are, thus leaving us with 7 * 12 = 84 total possibilities.

If the digits are 4 and 9:
There are 3*2 = 6 ways to place 4 and 9, and then 2*2 = 4 ways to fill the remaining digits with 1 and 0. Thus there are 6 * 4 = 24 ways total.

Thus for 2 non-zero/one digits we have 24 + 20 + 84 = 128

3 non-zero/one digits:

There are only two possibilities for this. Either 2,3,6 are the three digits or it is a perfect square (one of 4 or 9) combined with two of the same digit:

2,3,6: In this case, all 3 digits must be in the last 3 digit positions. Thus there are 3! = 6 arrangements. Since the last digit can either be 1 or 0, this brings it up to 6 * 2 = 12 possibilities.

Perfect square + two digits: This scenario is identical for 4 and 9. For each of these, there are 7 numbers where the two digits are not equal to the perfect square itself. Thus, since all non-zero/one digits have to be in the last 3 spots, there are 3C2 = 3 ways to arrange these digits. Additionally, the leading digit is either 1 or 0, which brings another 2 possibilities. Thus 3*2 = 6 possibilities. Since 7 digits satisfy this, this brings a total of 6*7 = 42 possibilities. If the pair of digits is equal to the perfect square (i.e. 444 or 999), then there are only 2 possibilities (if the first digit is 0 or 1). Thus there are 44 possibilities for each of 4 and 9, leading to 2 * 44 = 88 possibilities in total.

This means there are 100 total possibilities for 3 non-zero digits:

Thus in total, there are 15 + 48 + 128 + 100 = 291

Answer to F3: 291

F4

Alice and Bob are playing a game involving a standard deck of cards and n dice. Each "turn", Alice draws a random card from the deck, while Bob rolls the dice. If Bob's dice roll produces a number higher than the number on Alice's randomly drawn card, Bob wins. Otherwise, Alice shuffles the card back into her deck, and Bob moves on to the next dice. If Bob runs out of dice, he loses the game. Find the minimum value of n such that Bob is expected to win at least 90% of the time.

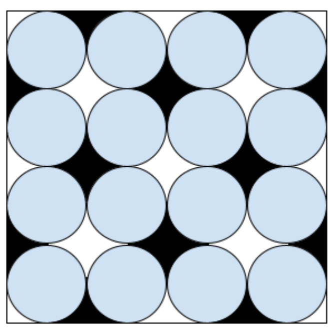
(For the sake of this question, A = 1, J = 11, Q = 12, and K= 13 in the deck of cards)

Solution: Consider that Bob has a $(\frac{1}{6})(\frac{0+1+2+3+4+5}{13}) = \frac{15}{78} = \frac{5}{26}$ chance of winning each turn. Thus, Alice has a $\frac{21}{26}$ chance of surviving each turn. Thus, in order for Bob to win 90% of the time, Alice must win less than 10% of the time. If n is the number of turns in the game, Alice wins $(\frac{21}{26})^n < 0.1$. By rearranging for n, we get $n > log_{\frac{21}{26}}(0.1) \approx 10.7812$. Thus, the minimum value of n is 11

Answer to F4: 11

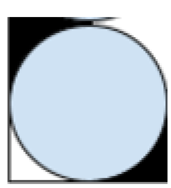
F5

Consider a chessboard pattern constructed by alternating colouring gaps between circles of radius 1 in a larger square. One such example of a pattern with 16 circles is provided



Consider a construction of this pattern with 100 circles and find the area of the black coloured region.

Solution: Consider dividing up the larger board into 100 individual squares each inscribing a circle. Consider that the area of the black region in each square will have half of the difference of the area between the square and the inscribed circle.



Thus, that would come out to $\frac{4-\pi}{2}$. Thus, the total area is $100 * \frac{4-\pi}{2} = 200 - 50\pi$

Answer to F5: $200 - 50\pi$

F6

In the milky way, there exists the planet Polaris, where all inhabitants come from two groups. The righteous Verits always tell the truth, while the evil Mendas always lie. You are given a collection of 10 inhabitants from the planet, all of which are either Verits or Mendas. You can ask any of these aliens any question that results in a yes or no answer. You are tasked with identifying which of the two groups each alien in the collection belongs to. Your predecessor was not very good at his job, and ended up taking 29 questions to identify the aliens. In how many fewer questions can you use that will guarantee that you identify all of the aliens?

Solution: Note that by asking the question "If I were to ask you if you were a Veritas, would you say yes?" will always result in a Yes answer from a Verits, and a No answer from any Mendas. Thus, 10 questions (asking each one of the aliens this exact question), would be sufficient.

Now notice that there are 2^{10} possibilities for the makeup of the aliens. Consider also that any question you ask results in a yes or no answer, meaning at most, you can gain at most, one bit of information, or in other words, at best with each question you can reliably eliminate half of the possibilities. Thus, since $2^9 < 2^{10}$, no strategy with 9 or fewer questions can reliably generate enough information, that for any number of Aliens you can guarantee that you can identify all of them with just 9 questions.

Thus, one can take 29 - 10 = 19 Fewer questions

Answer to F6: 19

Student Name:______________________________

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F7

Note that a polynomial is a mathematical expression containing variables and integer coefficients. For example, $x^2 + 3x + 7$ and $2x^4 + 7x + 1$ are examples of polynomials. For any given polynomial, we say its degree is equal to the highest exponent on any variable in the polynomial (the examples above have degree 2 and 4 respectively). Consider a polynomial with degree 2024. If you substitute $x = 0$, this polynomial is evaluated to 6. If you substitute $x = 1$ into the polynomial, you will get a non-zero value. The polynomial has only integer roots. There are two possible values for the coefficient of the x^{2022} term. Find the difference between them

Solution: Let the polynomial be P(x). Consider that the P(x) can be factored into $P(x) = (x - r_1)(x - r_2) ... (x - r_{2024})$ where $r_1, r_2, ... r_{2024}$ are the integer roots of the polynomial. Consider that since P(0) = 6, $r_1 r_2 ... r_{2024} = 6$, i.e. the product of the roots is 6. Also note that since P(1) is non-zero, 1 is NOT a root of P. Thus, we can dedcude from this that there are 2 possibile sets of roots for P. Either -1, -2, and -3, or -1,2 and 3. In both scenarios there are 2022 repeated roots of -1 along with the two other rroots. Thus we have either $(x + 1)(x + 1)(x + 1)... (x + 2)(x + 3)$ or $(x + 1)(x + 1)(x + 1)... (x - 2)(x - 3)$. In both scenarios, the coefficient of the x^{2022} term can be found by adding up the products of combinations of the roots. Note for both polynomials, there are 2022C2020 combinations that have a product of 6, and a singlke combination of product 1 (choosing both the unique non -1 roots). However, the difference is when one of the two unique roots is chosen. There are 2022C2021 = 2022 combinations for each of them, which has added 2022 * 3 + 2022 *2 = 10110 for the negative roots amd 2022 * -3 + 2022 * -2 = -10110 for the positive roots. Thus the difference is 10110- (- 10110) = 20220 in their coefficients of the x^{2022} term.

Answer to F7: 20220

Student Name:______________________________

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F8

Consider the base 2024 number 20242024 … 2024 (where the number is 2024 repeated 2024 times). Find the remainder of this number when it is divided by 17.

Consider that $2024 \equiv 1 \mod 17$. Thus we can also conclude that $2024^n \equiv 1^n \equiv \mod 17$. Note then that $x * 2024^n \equiv x \mod 17$. Therefore, we can note that 20242024 … 2024 has the same remainder when divided by 17 as the sum of the digits of 20242024 … 2024's remainder when divided by 17. Thus, since the sum of the digits of 2024 is 8, and thus the sum of the digits of 20242024 … 2024 is $8 * 2024 = 16192$. Thus note $16192 \equiv 8 \mod 17$. Thus, the answer is 8.

Answer to F8: 8