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Section E

E1.

Timmy is not doing too well in his math courses! He currently has a 60% average, and each of his tests in the course are weighed equally. After getting a 80% on his next test, he has a new average of 64%. Timmy has two tests remaining in the course. If he gets a 80% on one of the tests, what is the minimum grade he must get on the other in order to have a 70% average in the course?

Solution: Let n be the number of tests Timmy has taken, and x is the cumulative grade of Timmy. Notice that $\frac{x}{n}$ = 60. Since Timmy gets a 80% on his next test and his average goes up to 64%, we can write $\frac{x+80}{n+1}$ = 64. Thus, we can see that $x = 60n$ and then that $60n + 80 = 64n + 64$, rearranging to get $4n = 16$, $n = 4$. Thus, $x = 240$ and he will take 7 tests in total. Since he wants an average of 70 and he gets 80 on one of his next tests, if we let y be the test he gets on his next grade, we can write out the following equation.$\frac{320 + 80 + y}{7}$ = 70

$490 = 320 + 80 + y$

$y = 90$

Answer to E1: 90

E2

How many different ways are there to arrange the letters in "ILIKEMATH"?

Solution: Note that this is a simple permutation with repetition problem. Since the I is repeated, we can calculate the total amount of permutations as:

$\frac{9!}{2!} = 181440$

Answer to E2: 181440

E3

How many isosceles triangles with integer sides are there with perimeter less than 20?

Solution: Consider that since the side lengths must add up to less than 20. Consider the length of the two equal length sides as p, and the remaining side length as k.
p = 1: then k = 1 as any other value would violate triangle inequality
p = 2: then k = 1, 2, or 3 as higher values would violate triangle inequality
p = 3: then k = 1, 2, 3, 4, or 5 as higher values would violate triangle inequality
p = 4: then k = 1, 2, 3, 4, 5, 6, or 7 as higher values would violate triangle inequality
p = 5: then k = 1, 2, 3, 4, 5, 6, 7, 8 or 9 as higher values would violate triangle inequality
p = 6: then k = 1, 2, 3, 4, 5, 6, or 7 as higher values would have too large perimeters
p = 7: then k = 1, 2, 3, 4, or 5 as higher values would have too large perimeters
p = 8: then k = 1, 2, or 3 as higher values would have too large perimeters
p = 9: then k = 1 as higher values would have too large perimeters

Thus, in total there are 1 + 3 + 5 + 7 + 9 + 7 + 5 + 3 + 1 = 41

Answer to E3: 41

E4

Bob is playing darts on the following bullseye target, where each ring has “width” 1 and the center bullseye has radius of 1. Bob is equally as likely to hit any point on the target (he will also always hit the target), what is Bob’s average score per throw?

Calculate the area of each respective point section:

10: It is just a circle of radius 1, which has π area

7: It is the ring formed by the larger circle (which has radius 2) minus the smaller circle (radius 1). Thus the total area is 4π - π = 3π.

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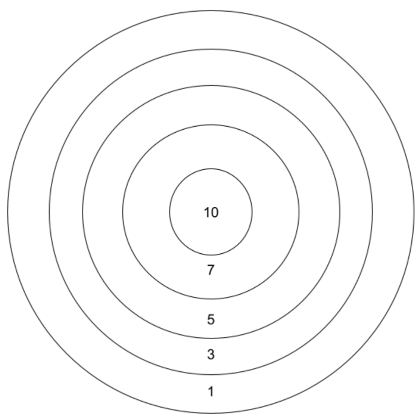
5: It is the ring formed by the larger circle (which has radius 3) minus the smaller circle (radius 2). Thus the total area is 9π - 4 π = 5π.

3: It is the ring formed by the larger circle (which has radius 4) minus the smaller circle (radius 3). Thus the total area is 16π -9 π = 7π.

1: It is the ring formed by the larger circle (which has radius 5) minus the smaller circle (radius 4). Thus the total area is 25π - 16π = 9π.

Since the total dartboard has area of 25π, the expected value of each hit is

$$E = \frac{10(\pi) + 7\,(3\pi) + 5(5\pi) + 3(7\pi) + 1(9\pi)}{25\pi} = \frac{(10 + 21 + 25 + 21 + 9)\pi}{25\pi} = \frac{86\pi}{25\pi} = \frac{86}{25} = 3.44$$



Answer to E4: $\frac{86}{25}$ or 3.44

E5

Jerry and Gary are playing a game. Jerry writes n integers less than 100 which are coprime to each other. Gary then must choose a prime number x randomly. If any of Jerry's numbers are divisible by x, Gary wins. Otherwise, Jerry wins. What is the minimum value of n such that Gary will always win.

Note that since Jerry must write n coprimes, if he has to write more numbers than there are primes less than 100, then there necessarily exists a number a for each prime p, where p | a. Thus, Gary will always win. If Jerry has to write exactly the number of primes less than 100 = n, then Jerry can write down 1 (which is coprime to all other numbers) and n -1 primes. Thus there is a chance that Gary chooses the only remaining prime, thus losing him the game. Thus, the minimum value of n, should be exactly 1 more than the number of primes less than 100, which is 26

Answer to E5: 26

E6 Note that the sum of the digits of 2024 is equal to 8. We call this number a "special" number. Find how many 4 digit special numbers there are.
(Reminder: Numbers such as 0135 and 0967 are not 4 digit numbers)

Solution:
Note the possible combination of digits that would sum up to 8 and group them by how many zeroes are present:
Case 1, 3 zeros: (8000), this has exactly 1 valid arrangement, as any other arrangement would cause it to not be a 4 digit number.
Case 2, 2 zeroes : There are actually 2 different subcases depending on if the non-zero digits are distinct or not

Case 2a, 2 distinct non-zero digits: There are a total of 3 ways (7100, 6200, 5300) to choose 4 digits with 2 that are 0 and 2 that are distinct non-zero digits. For each choice, there are 6 valid arrangements, leading to 18 total numbers.

Case 2b, 2 of the same non-zero digit: There is 1 way (4400) to choose in digits in this case, and there are 3 valid unique arrangements.

Thus in total, there are 21 valid numbers here.
Case 3, 1 zero: There are 2 subcases here depending on the number of distinct non-zero digits present in the number

Case 3a, 2 distinct non-zero digits: There are a total of 3 ways (6110, 4220, 3320) for this to happen. In each choice, there are 9 valid arrangements, leading to 27 total numbers.

Case 3b: 3 distinct non-zero digits: There are a total of 2 ways (5210, 4310) for this to happen. In each choice, there are 18 valid arrangements, leading to 36 total numbers.

Thus there are 63 valid numbers here.

Case 4, no zeroes: There are 3 subcases depending on the number of distinct digits present

Case 4a, 1 distinct digit: There is only a single valid number and arrangement here, which is 2222.

Case 4b, 2 distinct digits: There are a total of 2 ways to choose the digits (5111, 3311), and there are 4 possible arrangements in 5111, leading to 4 total numbers. For 3311, there are 6 possible arrangements. Thus in total, there are 10 valid numbers.

Case 4c, 3 distinct digits: There are a total of 2 ways to choose the digits (4211, 3221), and for 4211 and 3221, there are 12 valid arrangements each. Thus, in total there are 24 valid numbers.

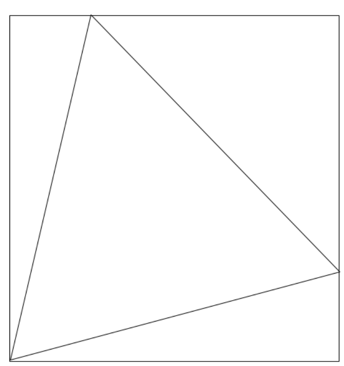
Thus in total for this case, there are 35 possible valid numbers.

Thus in total there are 35 + 63 + 21 + 1 = 120 ways.

Answer to E6: 120

E7

In the city of Waterloo, the city uses a special type of pipe, where a triangular prism with equilateral faces is then encased by a rectangular prism pipe with square faces. An example of a cross section of the pipes is provided below



If the side lengths of the square are equal to 1m, what is the total volume of a 100m length section of the smaller triangular pipe?

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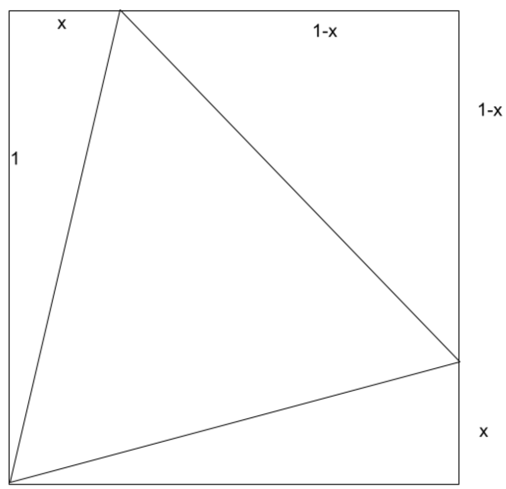
Solution:
Consider labelling one of the sections of the pipe as x, so that we can draw the following diagram.



Note that since we have an equilateral triangle, we know its side lengths are equivalent. By pythagorean theorem, we know two representations of the side length of the triangle. One is $\sqrt{x^2 + 1}$ and the other is $\sqrt{2}(1 - x)$. By setting these two as equal to one another, we can determine that $x^2 + 1 = 2(1 - x)^2$ or $x^2 + 1 = 2(1 - 2x + x^2) = 2 - 4x + 2x^2$ or $x^2 - 4x + 1 = 0$. Solving for this gets $x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$. However, as x < 1, we can conclude $x = 2 - \sqrt{3}$. Thus, the area of the triangle $1 - (2 - \sqrt{3}) - \frac{(\sqrt{3} - 1)^2}{2} =$ $1 - (2 - \sqrt{3}) - \left(2 - \sqrt{3}\right) = 2\sqrt{3} - 3$. Multiply this by 100 to get the volume of the pipe, which is equal to $200\sqrt{3} - 300$

Answer to E7: $200\sqrt{3} - 300$

E8: What is the smallest number n such that 2024 in base n is divisible by 11.
(Recall that for example, 2024 in base n is equal to $2n^3 + 0n^2 + 2n + 4$)

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Note $11 | 2n^3 + 2n + 4$. Thus we can state, $2n^3 + 2n \equiv 7 \bmod 11$. Also note that $n > 4$ since otherwise the 4 digit cannot exist.

We can now do some casework:

If $n = 5$, $2n^3 + 2n = 260 \equiv 7 \bmod 11$, which immediately confirms to us that $n = 5$.

Answer to E8: 5