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6 Section F

F1
If $2^x = 8^y$ what is the value of $\frac{x}{y}$?

Solution. If $2^x = 8^y$ then $2^x = (2^3)^y = 2^{3y}$, so $x = 3y$, which means $\frac{x}{y} = 3$.

Answer to F1: 3

F2
Each day, Sarah can either take the bus or a scooter to work, and each one costs money to use. In a 5 day work week from Monday to Friday, Sarah takes the bus three times and a scooter twice, for a total cost of \$19.25. On the weekend, she takes a scooter once and a bus once, for a total cost of \$8.00. How much more does the scooter cost to take than the bus?

Solution. Let the cost of the scooter be s and the cost of the bus be b. Then we have that $3b + 2s = 19.25$ and $b + s = 8$. Subtracting twice the second equation from the first, we get that $b = 19.25 - 16 = 3.25$. Substituting this back into the second equation, we get that $3.25 + s = 8$, and so $s = 4.75$. So the scooter costs $4.75 - 3.25 = \$1.50$ more than the bus.

Answer to F2: \$1.50

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F3

The following is the flag of Greece, which is drawn to scale. It has a 3:2 ratio between horizontal length and vertical length. What fraction of the flag is shaded?

Solution. We count that there are 9 horizontal stripes on the flag. Suppose that each stripe has length 1. Since the length to width ratio is $3:2$, the length of the flag is $\frac{27}{2}$. The total area of the flag is $\frac{243}{2}$. The four squares each have area $2 \times 2 = 4$. The two long shaded stripes each have area $\frac{27}{2}$. The three shorter stripes each have length $\frac{27}{2} - 5 = \frac{17}{2}$, and so this is each of their areas as well. The total area of the shaded regions is $4 \times 4 + 3 \times \frac{17}{2} + 2 \times \frac{27}{2} = \frac{137}{2}$. Then the fraction that is shaded is $\frac{137}{2} / \frac{243}{2} = \frac{137}{243}$.

Answer to F3: $\frac{137}{243}$

F4

A number between 1 and 200 ends in 7 in base 9, 5 in base 8, and 0 in base 7. In base 6, what does the number end in?

Solution. If a number ends in 0 in base 7, it is a multiple of 7. If it ends in 5 in base 8, it is 5 more than a multiple of 8. Checking multiples of 7 from 1 to 200, we find that $21, 77, 133$, and 189 are the only possibilities. We check the remainders of these four numbers when divided by 9 to get $3, 5, 7$, and 0 respectively. So our number is 133. To find its last digit in base 6, we compute the remainder when 133 is divided by 6 to get 1.

Answer to F4: 1

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F5
What is the side length of a regular tetrahedron whose volume and surface area are equal?

Solution. The faces of a tetrahedron are equilateral triangles. To find the height of an equilateral triangle with side length s, draw a line down the middle, and then use Pythagorean Theorem to get a height of $\sqrt{s^2-(s/2)^2}=\frac{\sqrt{3}}{2}s$. Then the triangle has area $\frac{1}{2}\times\frac{\sqrt{3}}{2}s\times s=\frac{\sqrt{3}}{4}s^2$. Since the tetrahedron has four faces, its surface area is $\sqrt{3}s^2$.

For volume, we want to know the height of the pyramid. This extends from the top point of the pyramid down to the middle of the bottom face. Connect the middle of this face to a corner, in order to get a right angled triangle standing upward, with hypotenuse s. It remains to find the distance from the corner of an equilateral triangle to its center. Draw a line through center from a vertex to the midpoint on the opposite side. Let the distance from the center to the vertex be x, and from the center to the midpoint of the opposite side y. Then from before, we have that $x+y=\frac{\sqrt{3}}{2}s$. Also, if we draw those lines from all three vertices to the opposite midpoints, we see triangles formed with side lengths of $\frac{1}{2}s$, y, and x, with x being the hypotenuse. By Pythagorean Theorem, $x^2=y^2+(s/2)^2=y^2+s^2/4$. Rearrange the first equation to get $y=\frac{\sqrt{3}}{2}s-x$, and substitute this into the second equation to get $x^2=(\frac{\sqrt{3}}{2}s-x)^2+s^2/4=\frac{3}{4}s^2-\sqrt{3}sx+x^2+s^2/4$. Rearranging, we get that $\sqrt{3}sx=s^2$, and so $x=s/\sqrt{3}$.

Moving back to the upright triangle with this new information, we get by Pythagorean Theorem that the height of the pyramid is $\sqrt{s^2-(s/\sqrt{3})^2}=\frac{\sqrt{2}}{\sqrt{3}}s$. The volume of a pyramid is given by base times height divided by 3, so in this case it is $\frac{1}{3}\times\frac{\sqrt{3}}{4}s^2\times\frac{\sqrt{2}}{\sqrt{3}}s=\frac{\sqrt{2}}{12}s^3$. If the volume and surface area are equal, then $\sqrt{3}s^2=\frac{\sqrt{2}}{12}s^3$. We know that s is not 0 since it is a side length, so $s=\frac{12\sqrt{3}}{\sqrt{2}}=6\sqrt{6}$.

Answer to F5: $6\sqrt{6}$

F6
How many six digit numbers consist only of digits 1, 2, and 3, and are divisible by 7?

Solution. The easiest way to solve this question is by using modular arithmetic (i.e. looking at remainders when divided by 7). We write $a \equiv b \mod 7$ to mean that a has a remainder of b when divided by 7. We have that $1 \equiv 1 \mod 7, 10 \equiv 3 \mod 7, 100 \equiv 2 \mod 7, 1000 \equiv 6 \mod 7, 10000 \equiv 4 \mod 7, 100000 \equiv 5 \mod 7$. The first three digits give us $100000a + 10000b + 1000c \mod 7$, where a, b, c can each be either $1, 2$, or 3. Similarly, the last three digits give us $100d + 10e + f \mod 7$, where d, e, f can be any of $1, 2, 3$. What remainder these sums have is summarized in the table below.

Sums of last three (fixing $f = 1$)	Sums of first three (fixing $c = 1$)
1 + 3 + 2 = 6	6 + 4 + 5 = 1
1 + 3 + 4 = 1	6 + 4 + 3 = 6
1 + 3 + 6 = 3	6 + 4 + 1 = 4
1 + 6 + 2 = 2	6 + 1 + 5 = 5
1 + 6 + 4 = 4	6 + 1 + 3 = 3
1 + 6 + 6 = 6	6 + 1 + 1 = 1
1 + 2 + 2 = 5	6 + 5 + 5 = 2
1 + 2 + 4 = 0	6 + 5 + 3 = 0
1 + 2 + 6 = 2	6 + 5 + 1 = 5

If $f = 2$, add 1 to each corresponding entry, and if $f = 3$, add 2. If $c = 2$, subtract 1, and if $c = 3$, subtract 2 (since $6 = -1 \mod 7$).

We are looking for the number of ways to combine an entry on the left with an entry on the right to reach a sum of 0 mod 7. This is computationally heavy, but thankfully we are allowed a calculator. On the left side, across the cases $f = 1, 2, 3$, there are four ways to make every number from 0 to 6 except for 5, where there are three ways. On the right side, across the cases of $c = 1, 2, 3$, there are four ways to make every number from 0 to 6 except for 2, where there are three ways. Since 5 and 2 combine together to make 0, we thus get that there are $6 \times 4^2 + 3^2 = 105$ valid numbers.

Note that there are many alternate solutions to this question, including ones using the divisibility rule for 7.

Answer to F6: 105

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F7

William rolls four 6-sided dice. He discards the lowest roll, and then sums the remaining three rolls. On average, what can he expect the sum to be?

Solution. The expected value of a die roll is the average over all possible rolls, that is $\frac{1+2+3+4+5+6}{6} = 3.5$. Then the expected sum of four rolls is $4 \times 3.5 = 14$. If there are 6^4 possible combinations of rolling four dice, then the sum of sums for all those combinations is $6^4 \times 14$. Now we want to subtract off 1 according to the number of combinations in which our lowest roll is a 1, subtract off 2 a number of times equal to the number of combinations in which our lowest roll is a 2, and so forth, in order to get the expect sum of the highest three numbers.

The probability that none of our rolls are 1 is $(\frac{5}{6})^4$, and so the probability of the lowest roll being a 1 is $1 - (\frac{5}{6})^4 = \frac{671}{6^4}$. The numerator of this expression gives us the number of combinations of four rolls where the lowest roll is a 1. Similarly, since the probability that none of our rolls is a 1 is $(\frac{5}{6})^4$ and the probability that none of our rolls is 1 or 2 is $(\frac{4}{6})^4$, the probability that the lowest roll is a 2 is $(\frac{5}{6})^4 - (\frac{4}{6})^4 = \frac{369}{6^4}$. Note that this is not in lowest terms: it is important that we write this as an expression with denominator 6^4 so that we can find the correct number of combinations. The numerator, 369, is the number of combinations of 4 rolls where the lowest roll is a 2.

Proceeding along these lines, the number of combinations where the lowest number is $3, 4, 5,$ and 6 respectively is $175, 65, 15,$ and 1 respectively. The expected sum of the highest three numbers is thus $\frac{6^4\times14-671\times1-369\times2-175\times3-65\times4-15\times5-1\times6}{6^4} = \frac{15869}{1296}$, which is approximately 12.24.

Answer to F7: $\frac{15869}{1296}$

F8

Alex and Brian are playing a game with a deck of cards. Each card has one of four suits (spades, clubs, hearts, or diamonds) and one of 13 ranks $(A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K)$, for a total of $13 \times 4 = 52$ cards in the deck. The game proceeds as follows: Alex draws n cards from the deck, and chooses one to place face down. He then arranges the remaining $n - 1$ cards in whatever order he wants, and then gives them to Brian. Brian then attempts to guess which card Alex has face down based only on which cards he received and the order that they are in. They both win if Brian guesses correctly, otherwise they lose. If they are allowed to agree on a strategy beforehand, what is the smallest value of n for which Brian can always correctly guess what Alex's face down card is?

Solution. Notice that if we have a strategy for some n, then we automatically have a strategy for $m > n$. Brian should just ignore the last $m - n$ cards he's passed and use the strategy for n on the remaining cards. This means that if some n is impossible, then any $m < n$ is impossible as well, so it suffices to find the value of n such that the game is impossible for $n - 1$ but possible for n.

It's not too hard to find a solution for $n = 6$. Pick some ordering on the 52 cards. Take any six cards $x_1, x_2, x_3, x_4, x_5, x_6$. We will take x_1 to be the facedown card, and suppose that $x_2 < x_3 < x_4 < x_5 < x_6$. Suppose that in our ordering, x_1 is card n. To communicate this, place an order on the $5! = 120$ permutations of x_2, x_3, x_4, x_5, x_6. (i.e. if we number them from least to greatest, then the order is $12345, 12354, 12435, \ldots, 54321$). Put them in the order corresponding to permutation n, and pass them to the second person.

How about $n = 5$? This one is trickier. There are quite a few strategies that work, but I think this one is the most elegant. Since there are four suits in the deck (spades, clubs, hearts, and diamonds), and five cards, there must be one suit with at least two cards. Pick two cards with the same suit, and call them x_1 and x_2. Call the other three cards x_3, x_4, and x_5. Put x_1 face down. The idea is to give Brian x_2 first, in order to communicate the suit to him. We have three cards remaining, and we can use permutations of them to communicate the $3! = 6$ numbers between 1 and 6. The problem is that we have to somehow distinguish between the 13 ranks of this suit. Our saving grace is that fact that we didn't necessarily need to have put x_1 face down and used x_2 to communicate its suit - we could instead put x_2 face down and use x_1 to communicate the suit. Write the ranks $A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K$ in a circle. Notice that any pair can have a distance of at most 6 in the shortest of the two directions. So pick the one in the pair (x_1, x_2) so that travelling clockwise, it takes us at most 6 steps to get to the other, and put it face down. Now x_3 through x_5 can be used to communicate whether to take $1, 2, 3, 4, 5$, or 6 steps along the circle, and so we have a winning strategy.

This strategy was very precise, and so it'd be a reasonable guess that $n = 4$ is impossible. For completeness' sake, we prove that it isn't possible. Suppose that it were. One might be tempted to then define a hypothetical strategy function $f(x, y, z)$ that takes in an ordered triple (x, y, z) of cards and outputs a card, and has the property that for any set of four cards, there is some ordering of x_1, x_2, x_3, x_4 satisfying $x_1 = f(x_2, x_3, x_4)$. This turns out to be far harder than anticipated, and I can't think of a way to make such a proof work. The key is to realise that the ordered triple tells us much more information than just what the face down card is: it gives us the original set of four cards! So we can view a winning strategy as being a surjective function from ordered triples of cards to sets of four cards. Notice that there are $52 \times 51 \times 50$ ordered triples of cards but $\binom{52}{4}$ sets of four cards. We have that

$$\binom{52}{4} = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2} > 52 \times 51 \times 50.$$

Our function can't possibly hit every set of four cards, a contradiction. We conclude that no such strategy exists for $n = 4$, and so the answer is 5.

Answer to F8: 5