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5 Section E

E1
What is the measure of each angle of a regular hexagon?

Solution. One can connect opposite vertices of the hexagon to get six equilateral triangles with a common point at the centre. Each angle of the hexagon consists of the angles of two equilateral triangles. Since the angles of a triangle sum to 180°, each angle is $180/3 = 60^\circ$, and so two of these gives 120°.

Answer to E1: 120°

E2
We have that $\frac{5}{64} = 0.078125$, and so the 4th digit of the decimal expansion of $\frac{5}{64}$ is a 1. What is the 2022nd digit of the decimal expansion of $\frac{1}{7}$?

Solution. Using a calculator, we get that $1/7 = 0.\overline{142857}$. These numbers repeat every six digits, and so the only important detail is the remainder of 2022 when divided by 6. As it turns out, $2022 = 6 \times 337$, and so the digit must be a 7.

Answer to E2: 7

E3
In a list of the natural numbers $1, 2, 3, 4, 5, \cdots$, which number contains the 111th occurrence of the digit 1?

Solution. There are $10^2 = 100$ numbers containing either 1 or 2 digits (leading zeroes don't count as separate digits) but we will write 3 as 03, for example. Of these, there are 10 numbers that have 1 as the first digit, 10 that have 1 as the second digit, and we have counted the number 11 twice, which has two occurrences of 1, so we don't have to worry about double-counting. This gives 20 occurrences from 1 to 99. Now for three digits, the block from 100 to 109 contains one 1 per number, except 101 which contains two. There are 10 numbers in this block, so this is 11 occurrences total. The same holds for the blocks $120 - 129, 130 - 139$, and so forth. The block from 110 to 119 contains two 1's per number, except 111 which contains three. There are again 10 numbers, so this is $2 \times 10 + 1 = 21$. Thus after 119 we have seen $20 + 11 + 21 = 52$ ones. We have that $(111 - 52)/11 \approx 5.36$, so we run through five more sets of ten digits to get to 169. This gives 107 occurrences. Then 170 contains occurrence 108, 171 contains occurrences 109 and 110, and 172 contains occurrence 111.

Answer to E3: 172

E4

What digit does the number $5^{2022} + 3 \times 6^{2023}$ end in?

Solution. Notice that $5^2 = 25, 5^3 = 125, 5^4 = 625$, and so any power of 5 always ends in 5. Similarly, $6^2 = 36, 6^3 = 216$, and so forth, so any power of 6 always ends in 6. Then 3×6^{2023} will end in the same digit as 3×6, which is 8. Thus $5^{2022} + 3 \times 6^{2023}$ ends in the same digit as $5 + 8 = 13$, which is 3.

Answer to E4: 3

E5

The *serpent value* of an integer is calculated by alternately inserting '-' and '+' signs between the digits, with '-' coming first. For example, the serpent value of 1427 is $1 - 4 + 2 - 7 = -8$, and the serpent value of 9 is just 9. What is the sum of all serpent values of numbers from 1 to 999?

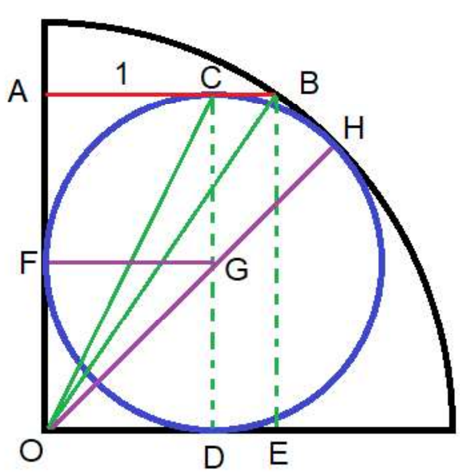
Solution. Among one digit numbers, the serpent value is just the number itself. This gives a total of $1 + 2 + \cdots + 9 = 45$. It would be wise to remember this total since it shows up repeatedly later. Now for a two digit number AB, the serpent value is $A - B$. Each value of A appears for 10 different B, while each non-zero value of B appears for 9 different A. So the sum of all two digit serpent values is $10(1+2+\cdots+9) - 9(1+2+\cdots+9) = 1+2+\cdots+9 = 45$. For a three digit number ABC, the serpent value is $A - B + C$. Each value of A appears for $10 \times 10 = 100$ different (B, C) pairs, while each value of B appears for $9 \times 10 = 90$ different (A, C) pairs and each C appears for $9 \times 10 = 90$ different (B, C) pairs. This totals $100(1+2+\cdots+9) - 90(1+2+\cdots+9) + 90(1+2+\cdots+9) = 100(1+2+\cdots+9) = 100(45) = 4500$. Thus in total we have $4500 + 45 + 45 = 4590$.

Answer to E5: 4590

E6

A circle is inscribed within a larger quarter circle such that it is tangent to all three sides of it. The line segment tangent to the circle and parallel to the bottom of the quarter circle, with endpoints on the other two sides of the quarter circle, has length 1. What is the radius of the inscribed circle?

Solution. Construct the following diagram.



Student Name: ______________________________
Please write your name on ***every*** page.

Let r be the radius of the inscribed circle, and R the radius of the quarter circle. By Pythagorean Theorem in triangle OBE, $1^2 + (2r)^2 = R^2$, or $1 + 4r^2 = R^2$. Since OA is tangent to the inscribed circle at F, FG is perpendicular to OA, and similarly GD is perpendicular to OE. Since they meet at the centre of the circle, OFGD is a square with side length r. Then $OG^2 = r^2 + r^2 = 2r^2$, so $OG = \sqrt{2}r$. Then $R = OH = OG + GH = \sqrt{2}r + r = (1 + \sqrt{2})r$. Squaring both sides, $R^2 = (1+\sqrt{2})^2 r^2 = (3+2\sqrt{2})r^2$. Substituting this into the first equation, $1 + 4r^2 = (3+2\sqrt{2})r^2$, or $1 = (2\sqrt{2} - 1)r^2$. Thus $r = 1/\sqrt{2\sqrt{2}-1}$. This is as simplified as it gets.

Answer to E6: $1/\sqrt{2\sqrt{2}-1}$

E7

Ben wakes up on Monday and walks to school, only to find that a flock of geese have moved onto the path he takes, and will not be leaving until the beginning of next week. Fortunately, the geese will let him pass for the day if Ben bribes them with n treats, but only if n is a positive integer, and shares no factors other than 1 in common with any number of treats he has bribed them with on previous days. If Ben wants to go to school five days in a row (Monday through Friday), has never bribed the geese before Monday, and never has to bribe the geese again after Friday, what is the minimum number of treats that Ben must give the geese over the course of the week?

Solution. This one is somewhat of a trick question. One might think of using prime numbers, since these have no factors in common with one another, but careful examination reveals that it isn’t forbidden to use the number 1 itself. Indeed, 1 shares no factors other than 1 in common with 1, and so we can simply give 1 treat each day for five days, giving 5 treats in total.

Answer to E7: 5

E8

All permutations of the string GRANDRIVER are listed in alphabetical order (for example, ADEGINRRRV is the first permutation on the list, while ADEGINRRVR is second. In what position does the string GRANDRIVER itself appear on the list?

Student Name: ______________________________

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Solution. Before the entries with first letter G, we have all the entries with first letter A, D, or E. How would we count the entries with first letter A? We can start by treating the three R's as distinct letters R_1, R_2, R_3. Then all letters are distinct, and so this is just permutations of 9 distinct elements, which can be done in 9! ways. We have actually counted each valid ordering 3! times, since we could swap the orderings of R_1, R_2, R_3 in any way we like: without the subscripts they just look like R and so the permutations are the same. Thus $3 \times 9!/3!$ strings appear before those beginning with G.

Now that we have locked G into place, we can repeat this process on the rest of the letters. We have that A, D, E, I, N all appear before R as a second letter. This gives $5 \times 8!/3!$ strings. With GR locked in, note that we now only have two copies of R, and so we should divide by 2! in the future instead of 3!. We summarise the rest of the steps in the table below. A dash indicates that a letter comes first out of the letters remaining, and thus we may add it without worrying.

letter	G	R	A	N	D	R	I	V	E	R
previous	$3 \times 9!/3!$	$5 \times 8!/3!$	-	$3 \times 6!/2!$	-	$2 \times 4!/2!$	3!	$2 \times 2!$	-	-

From here, one can use a calculator to sum the entries in the table to get 216154. But this is how many entries come before GRANDRIVER. If we want the entry number of GRANDRIVER itself, we must add 1 to this, for a total of 216155.

Answer to E8: 216155