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6 Section F

F1

Find all values of x such that $1/(x-3) - 3/(x-1) + 3/(x+1) - 1/(x+3) = -3$.

Solution. We have that $\frac{3}{x+1} - \frac{3}{x-1} = \frac{3(x-1)-3(x+1)}{x^2-1} = \frac{-6}{x^2-1}$ and $\frac{1}{x-3} - \frac{1}{x+3} = \frac{(x+3)-(x-3)}{x^2-9} = \frac{6}{x^2-9}$. Adding these in turn gives the equation $\frac{-6(x^2-9)+6(x^2-1)}{(x^2-9)(x^2-1)} = \frac{48}{(x^2-9)(x^2-1)} = -3$, meaning $\frac{16}{(x^4-10x^2+9)} = -1$. Then $16 = -(x^4 - 10x^2 + 9)$, and so $x^4 - 10x^2 + 25 = 0$. This factors as $(x^2-5)^2 = 0$ which means that $x^2 = 5$, and hence $x = \pm\sqrt{5}$.

Answer to F1: $x = \pm\sqrt{5}$

F2

The surface area of a cylinder’s curved surface is 7 times larger than the surface area of one of its bases. What is the ratio of the cylinder’s radius to the cylinder’s height? (Write the ratio as a fraction in the form of $\frac{a}{b}$, with a and b in lowest terms.)

Solution. Let the cylinder have radius r and height h. The area of the curved surface is $2\pi rh$, and the area of one of the bases is πr^2. But we have that $2\pi rh = 7\pi r^2$, and so $\frac{2}{7} = \frac{\pi r^2}{\pi rh} = \frac{r}{h}$.

Answer to F2: $\frac{2}{7}$

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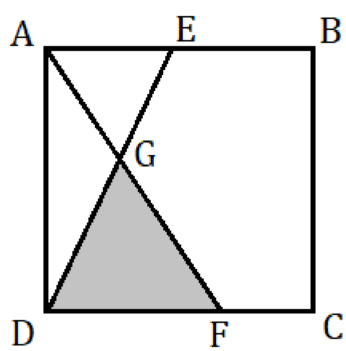
F3

In the diagram below, ABCD is a square with side length 6. AF and DE are drawn and intersect at G such that $DF = 4$ and $AE = 3$. Find the area of the shaded triangle $\triangle DFG$.



Solution. We can easily compute the area of $\triangle ADF$ as 12 and the area of $\triangle ADE$ as 9. Then since they have $\triangle ADG$ in common, it follows that the area of $\triangle DFG$ is 3 larger than the area of $\triangle AEG$. Let the height of the former be x and the latter be y. Then we know that $x + y = 6$ (from the side length of the square), and from the areas we get that $\frac{3x}{2} + 3 = \frac{4y}{2}$, meaning $3x + 6 = 4y$. Rearrange the first equation for $x = 6 - y$ and substitute into the second equation to get $3(6 - y) + 6 = 4y$, or that $7y = 24$, so that $y = \frac{24}{7}$. Then the shaded region has area $\frac{4y}{2} = 2y = \frac{48}{7}$.

Answer to F3: $\frac{48}{7}$

F4

The planet Mathzorg uses a different number system, which they call the Mathzorg system, compared to Earth's base 10 number system. The number 121 in the Mathzorg system is the number 16 in base 10. What is the Mathzorg number 2020 in base 10?

Solution. Let b be the Mathzorg base. Then we have that $b^2 + 2b + 1 = 16$. Factoring the left side gives $(b + 1)^2 = 16$, and so $b + 1 = 4$ since bases are positive, meaning $b = 3$. Then 2020 in base 3 corresponds to $2 \times 3^3 + 2 \times 3 = 54 + 6 = 60$.

Answer to F4: 60

F5

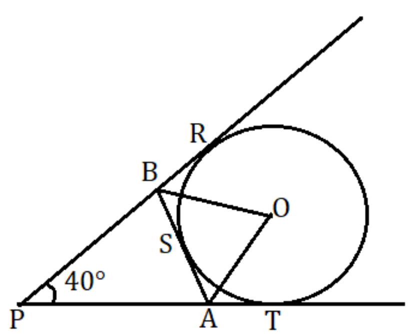
Jose wants to take his water bottle with him on a hike. It is a cylinder with diameter 4 cm and height 20 cm, with an opening on the top of diameter 2 cm. The issue is that Jose's water bottle does not have a cap, and the holder on his backpack holds it sideways, so that the opening is perpendicular to the ground. With this in mind, how high can he fill his water bottle when placed right side up so that no water will spill when it is held sideways?

Solution. Consider a cross section of the bottle. In order for no water to spill when the bottom is held perpendicular, the water line must lie below the bottom of the smaller circle (the opening). We have that the area of the large semicircle is $\frac{1}{2}\pi 2^2 = 2\pi$. Since the triangles are $1 : \sqrt{3} : 2$ triangles, the two cones below the diameter have angle 30°, and so their area is $2\frac{30}{360}\pi 2^2 = \frac{2\pi}{3}$. The area of the triangle is then $\frac{2\sqrt{3}}{2} = \sqrt{3}$, and so the shaded area is $2\pi - (\frac{2\pi}{3} + \sqrt{3}) = \frac{4\pi}{3} - \sqrt{3}$. Multiplying this by the length of the bottle which is 20, we find that the total volume of water is $20(\frac{4\pi}{3} - \sqrt{3})$. If the bottle is held upright, this forms a cylinder with height h, and hence volume $\pi 2^2 h = 4\pi h$. Equating volumes, $4\pi h = 20(\frac{4\pi}{3} - \sqrt{3})$, and so $h = \frac{20}{3} - \frac{5\sqrt{3}}{\pi}$.

Answer to F5: $\frac{20}{3} - \frac{5\sqrt{3}}{\pi}$

F6

In the diagram below, O is a circle and triangle PAB is formed by three tangents to O. What is the measure of $\angle AOB$?



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Solution. Connect OR and OT. Then $OR \perp BR$, $OT \perp AT$, and $OS \perp AB$ by properties of tangent lines. Thus $\triangle BRO$, $\triangle OAT$, $\triangle BSO$, and $\triangle SOA$ are right angled. Further, $OT = OS$ since they are both radii, and $AS = AT$ since they are both tangents to a circle from a common point. Then $\triangle SOA = \triangle OAT$. By a similar argument, $\triangle BSO = \triangle BRO$. Let $\angle BOS = \angle BOR = \alpha$ and $\angle AOT = \angle AOS = \beta$. Then $\angle SBR = 2(90^\circ - \alpha)$ and $\angle SAT = 2(90^\circ - \beta)$. Then $\angle PBA = 2\alpha$ and $\angle PAB = 2\beta$. Thus $40^\circ + 2(\alpha + \beta) = 180^\circ$, or $\alpha + \beta = 70^\circ$. Therefore $\angle AOB = 70^\circ$.

Answer to F6: 70°

F7

Find the greatest integer less than $\frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \ldots + \frac{1}{1000^{2/3}}$.

Solution. For any value of k we can write
$\frac{1}{k^{2/3}} = \frac{3}{k^{2/3}+k^{2/3}+k^{2/3}} > \frac{3}{k^{2/3}+k^{1/3}(k+1)^{1/3}+(k+1)^{2/3}} = \frac{3((k+1)^{1/3}-k^{1/3}}{(k+1)-k} = 3((k+1)^{1/3} - k^{1/3})$.
By a similar argument, we have that $\frac{1}{k^{2/3}} < 3(k^{1/3} - (k-1)^{1/3})$. Upon summing this from $k = 1$ to 1000, our bounds telescope and we get that $28 > 1 + 3(10-1) > 1 + 3(1000^{1/3} - 1^{1/3}) > 1000^{1/3} \frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \ldots + \frac{1}{1000^{2/3}} > 3(1001^{1/3} - 1^{1/3}) > 3(10-1) = 27$. This proves that 27 is the largest integer below this sum.

Answer to F7: 27

F8

Garrett is summing some series of numbers for fun. An **arithmetic sequence** is a sequence with an initial term a and common difference d between terms, so that the sequence looks like $a, a+d, a+2d, \ldots, a+(n-1)d$ after n terms.

An **arithmetic series** is the sum of an arithmetic sequence. In general, we have that

$$a + (a+d) + (a+2d) + \ldots + (a+(n-1)d) = \frac{n}{2}[2a + (n-1)d].$$

A **geometric sequence** is a sequence with an initial term a and common ratio r between terms, so that the sequence looks like $a, ar, ar^2, \ldots, ar^{n-1}$ after n terms.

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A **geometric series** is the sum of a geometric sequence. In general for $r \neq 1$, we have that

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

Garrett is bored of these sequences, and seeks to combine them to achieve something more interesting. Starting with an initial term a (not necessarily an integer), he multiplies by 2 and then adds a common difference d (again, not necessarily an integer) to get the second term. He then takes this term and multiplies by 2 again, and then adds d again to get the third term. He continues this until he has n terms, and then he adds all of those terms together.

Upon doing this with a certain a and d, he notices a few interesting coincidences:

- $a + d$ is an integer
- the series sums to 0
- nd is a perfect number, which means that the sum of the divisors of nd (not including nd itself) equals nd. Note that a perfect number can be written in the form $2^{p-1}(2^p - 1)$ for a prime number p, but not every such product is perfect (ex: $2^{10}(2^{11} - 1) = 2096128$ is not perfect).

Find the largest $d < 100$ for which this is possible.

Solution. Ignoring the criterion that $r = 2$, for general r this new sequence looks like
$t_1 = a$
$t_2 = ar + d$
$t_3 = (ar + d)r + d = ar^2 + dr + d$
$\vdots$
$t_n = ar^{n-1} + dr^{n-2} + \ldots + dr^2 + dr + d$
Then the sum of this series is
$S = t_1 + t_2 + \ldots + t_n = (a + ar + ar^2 + \ldots + ar^{n-1}) + nd + (n-1)dr + \ldots + 2dr^{n-3} + dr^{n-2}$
$= a(\frac{r^n-1}{r-1}) + (d + dr + \ldots + dr^{n-2}) + (d + dr + \ldots + dr^{n-3}) + \ldots + (d + dr) + d$
$= a(\frac{r^n-1}{r-1}) + d(\frac{r^{n-1}-1}{r-1}) + d(\frac{r^{n-2}-1}{r-1}) + \ldots + d(\frac{r^2-1}{r-1}) + d(\frac{r-1}{r-1})$
$= a(\frac{r^n-1}{r-1}) + \frac{d}{r-1}(r^{n-1} + r^{n-2} + \ldots + r^2 + r - (n-1))$
$= a(\frac{r^n-1}{r-1}) + \frac{d}{r-1}(r^{n-1} + r^{n-2} + \ldots + r^2 + r + 1 - n)$
$= a(\frac{r^n-1}{r-1}) + \frac{d}{r-1}(\frac{r^n-1}{r-1} - n)$
$= (a + \frac{d}{r-1})(\frac{r^n-1}{r-1}) - \frac{dn}{r-1} = 0$
Then $(a + \frac{d}{r-1})(\frac{r^n-1}{r-1}) = \frac{dn}{r-1}$
$(a + \frac{d}{r-1})(r^n - 1) = dn$, and letting $r = 2$ we then have $(a + d)(2^n - 1) = nd$. But nd is a perfect number, and hence can be written as $2^{p-1}(2^p - 1)$ for some p prime. Thus $a + d = 2^{n-1}$ and n is a prime. We can this reduce the problem to casework, considering the smallest primes.

- If $n = 2$, then $2^{2-1}(2^2 - 1) = 2(3) = 6$ is perfect, with $d = 3$ and then $a + d = 2^{2-1} = 2$ gives $a = -1$.
- If $n = 3$, then $2^{3-1}(2^3 - 1) = 4(7) = 28$ is perfect, with $d = \frac{28}{3}$ and then $a + d = 2^{3-1} = 4$ gives $a = -\frac{16}{3}$.
- If $n = 5$, then $2^{5-1}(2^5 - 1) = 16(31) = 496$ is perfect, with $d = \frac{496}{5} = 99.2$ and then $a + d = 2^{5-1} = 16$ gives $a = -\frac{416}{5}$.

Since the value of d increases with n and we are already near 100, clearly this is the largest possible d given the restrictions.

Answer to F8: $\frac{496}{5}$