Student Name: ______________________________

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5 Section E

E1

Isa is a fan of agriculture, and wants to build a fence to contain her cows. She is also a fan of shapes, and wants to either build the fence in an equilateral triangle or a square. She has 12 metres of fence to work with, and wants to use it all. How much larger is the area enclosed by her square fence idea than that which is enclosed by her equilateral triangle fence idea?

Solution. A square with perimeter 12 has side length $12/4 = 3$, and hence area $3 \times 3 = 9$. An equilateral triangle with perimeter 12 has side length 4. Then by drawing the height of the triangle, using Pythagorean theorem we get that the height is $\sqrt{4^2 - 2^2} = 2\sqrt{3}$. Then the area of the triangle is $4 \times 2\sqrt{3}/2 = 4\sqrt{3}$. Therefore the difference in areas is $9 - 4\sqrt{3}$.

Answer to E1: $9 - 4\sqrt{3}$

E2

What is the value of the product $(1 - \frac{1}{2}) \times (1 - \frac{1}{3}) \times \cdots \times (1 - \frac{1}{2019})$?

Solution. We can rewrite this as $\frac{1}{2} \times \frac{2}{3} \times \cdots \times \frac{2018}{2019}$. Each numerator and denominator will cancel except for the first and last respectively, giving us a final result of $\frac{1}{2019}$.

Answer to E2: $\frac{1}{2019}$

E3

Let $f_n(x)$ denote the function f applied n times to value x. For example, $f_4(x) = f(f(f(f(x))))$. Let $g(x) = x + 2$. Find the value of $g_{673}(673)$.

Solution. We have that $g(g(673)) = 673 + 2 + 2$, $g(g(g(673))) = 673 + 2 + 2 + 2$, and so in general $g_n(673) = 673 + 2n$. Then $g_{673}(673) = 673 + 2(673) = 2019$.

Answer to E3: 2019

E4

A bag contains 3 red, 2 blue, and 5 green marbles. Bob draws marbles from the bag until either all the red marbles or all the blue marbles have been drawn. What is the probability that the last marble drawn is blue?

Solution. The five non-green marbles must be drawn in some order. There are $\binom{5}{2} = 10$ possible arrangements of these, and fixing a red marble last in the order so that we draw the last blue one first, there are then $\binom{4}{2} = 6$ ways to arrange this. Thus our probability is $\frac{6}{10} = \frac{3}{5}$.

Answer to E4: $\frac{3}{5}$

E5

A cube has side length 4. A cylindrical hole of radius 1 is drilled through the centre of a face of the cube straight through to the opposite face. What is the surface area of the resulting solid?

Solution. The cube itself has a surface area of $6 \times 4 \times 4 = 96$. Drilling a cylindrical hole through the cube will remove the surface area corresponding to the ends of the cylinder, while adding a surface on the inside. The surface removed is $2\pi(1)^2 = 2\pi$ and the surface added is $2\pi(1)(4) = 8\pi$, resulting in a net increase of 6π. Thus the resultant surface area is $96 + 6\pi$.

Answer to E5: $96 + 6\pi$

E6

Five friends, Alice, Bob, Carl, David, and Earl, want to go see a movie at their local theater. Some of the friends have preferences to their seating plan:

- Carl wants to sit at the very left or right end of all the other friends.
- Alice and David prefer to sit beside each other.

How many different seating arrangements can there be if all of the preferences are met?

Solution. Fix Carl as sitting in the leftmost seat for now. Any valid arrangement where he is on the far right is simply a reflection of one where he is on the far left, so we just need to remember to multiply by 2 at the end. Now for the 4 other friends, Alice and David want to sit next to each other, so we treat them as one entity which itself has two possible arrangements. Then there are 3 ways to place the Alice-David pair, and for each of these there are 2 arrangements depending upon who is on the left. Then there are 2 ways to place Bob and then Earl goes in the only remaining place. Thus in total we have $2 \times 3 \times 2 \times 2 \times 1 = 24$ valid arrangements.

Answer to E6: 24

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E7

A number is called twoodd if it has exactly 2 different odd factors. For example, 6 is twoodd as it has only 1 and 3 as its odd factors, but 18 is not twoodd as it has 1, 3, and 9 as odd factors. How many numbers between 1 and 100 are twoodd?

Solution. The prime factorization of such a number looks like $2^a p$, $a \geq 0$ where p is an odd prime. For $p = 3$, a can range from 0 to 5. For $p = 5$, a can range from 0 to 4. For $p = 7, 11$, a can range from 0 to 3. For $p = 13, 17, 19, 23$, a can range from 0 to 2. For $p = 29, 31, 37, 41, 43, 47$, a can range from 0 to 1, and for $p = 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$, a can only be 0. This is a total of $6 + 5 + 4 \times 2 + 3 \times 4 + 2 \times 6 + 10 = 53$.

Answer to E7: 53

E8

Noah and Ben are playing a dice game. First Noah rolls a twenty sided die with numbers from 1 to 20. If the result is 13 or higher, then he is allowed to roll two six sided dice with numbers from 1 to 6, and sum them to determine how many points he gets. Ben opts to make the rules slightly more interesting. He will give Noah 10 extra points automatically whenever Noah rolls the six sided dice but Noah will only get to roll them if the result on the twenty sided die is 18 or higher instead of 13 or higher. How many fewer points can Noah expect to earn under the new system than under the old one?

Solution. Under the old system, Noah has a $\frac{8}{20} = \frac{2}{5}$ chance of rolling 13 or higher on a twenty sided die. Rolling two dice is an independent process, so the expected value of both of the six sided dice is twice the expected value of a single six sided die, which is $\frac{7}{2}$, so we expect a sum of 7. This totals $\frac{14}{5}$.

Under the new system, Noah now has a $\frac{3}{20}$ chance of rolling 18 or higher on a twenty sided die, and the expected value of two dice plus the ten bonus points is $10 + 7 = 17$. This gives us an expected value of $\frac{51}{20}$.

Overall, $\frac{14}{5} - \frac{51}{20} = \frac{56}{20} - \frac{51}{20} = \frac{5}{20} = \frac{1}{4}$ fewer points.

Answer to E8: $\frac{1}{4}$