Section F

F1
Let x be the unique positive real solution to $2^x = \frac{8}{x}$. What is x?

Solution. This can be solved by guess and check given the number must be a low number for exponentiation to equal 8.

The equation can be re-arranged as follows:

- $2x = \frac{8}{x}$
- $x \times 2x = 8$

The cube root (3rd root) of 8 is 2, which is also the base of the exponentiation. In this case, we see that when $x = 2$ the equation would hold.

Answer to F1: 2

F2
How many words can you make using all of the letters in CANADA? (The words formed do not need to be English words; for example, ACNDAA is a word.)

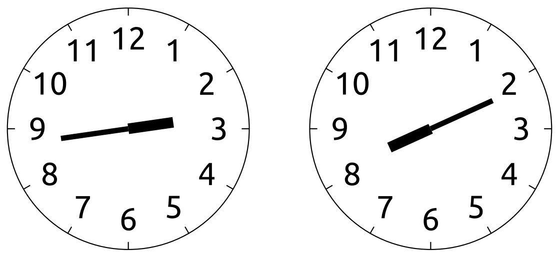
Solution. In total, there are 6 letters in CANADA, so the total number of combinations possible (including repetition) is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ (i.e. the first letter of the word can be any of the six letters, the second letter of the word can be any of the remaining five letters not already used) or $6! = 720$.

As the letter A is repeated 3 times in CANADA, the repetitions must be excluded in order to arrive at the unique number of words. Thus, $720/(3 \times 2 \times 1) = 120$ or $720/3! = 120$ (i.e. logic being that wherever the 3 A's are, the first slot could be any of the 3 A's, second slot could be any of the remaining 2 A's, and the last slot is the A that's not already used. However, all these would yield the same word no matter which A is used in which slot, and thus must be excluded).

Answer to F2: 120

F3

The drawing below shows analog clocks at around 2:44 and 8:11. At these times, the hour and minute hands are pointing in opposite directions. Between 1:00 AM and 11:59 AM on any day, how many times do hour and minute hands point in opposite directions?



Solution. The hour and minute hands of a clock point in opposite directions 11 times between 12:00 AM and 12:00 PM at points in time exactly $\frac{12}{11}$ hours apart. Only one of these points occurs outside the range of 1:00 AM to 11:59 PM (that point occurs between 12:00 AM and 1:00 AM, excluding 1:00 AM).

Answer to F3: 10

F4

How many distinct integer solutions (p, q) are there to this equation?

$$p^2 + 3pq + 2q^2 - p - q = 41$$

Solution. Our strategy is to *factor* the equation. We proceed as follows:

$$\begin{aligned} p^2 + 3pq + 2p^2 - p - q &= 41 \\ (p+q)(p+2q) - (p+q) &= 41 \\ (p+q)(p+2q-1) &= 41 \end{aligned}$$

Observe that the only ways to write 41 as the product of 2 integers are:

$$41 \times 1 \quad 1 \times 41 \quad -41 \times -1 \quad -1 \times 41$$

Each factorization gives a potential solution for (p, q). For example, for 41×1, we get

$$\begin{cases} p + q = 41 \\ p + 2q - 1 = 1 \end{cases} \implies \begin{cases} p = 80 \\ q = -39 \end{cases}$$

It's not hard to check that each of the other three factorizations gives a different integer solution for (p, q). So there are 4 solutions in total.

Answer to F4: 4

F5
What is the maximum possible value of $3x - y + z$ if x, y, z are real numbers such that $x \geq 0, y \geq 0, z \geq 0, x + 2y = 5$, and $x + y + z = 7$?

Solution. We will solve for the other two variables in terms of y. (Solving in terms of x or z will work too.) From $x + 2y = 5$, we get

$$x = 5 - 2y.$$

From $x + y + z = 7$, we get

$$z = 7 - x - y = 7 - (5 - 2y) - y = 2 + y.$$

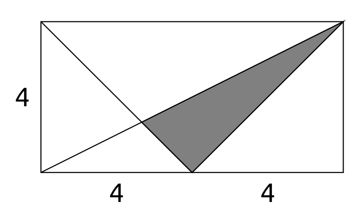
Therefore,

$$3x - y + z = (5 - 2y) - y + (2 + y) = 17 - 6y.$$

Since $y \geq 0$, this expression is maximized when $y = 0$, and its maximum value is 17. Finally, we need to check that if $y = 0$, then both x and z are non-negative too. If $y = 0$ then $x = 5$ and $z = 2$, so we're good. So the answer is 17.

Answer to F5: 17

F6
Find the area of the shaded triangle.



Solution. Label the vertices of the rectangle from the top-left corner in clockwise order A to D, let E be the midpoint of CD, and F be the intersection of AE and BD.

$\triangle ABF$ and $\triangle EDF$ are similar such that $\frac{AF}{EF} = 2$, so $EF = \frac{AE}{3} = \frac{4\sqrt{2}}{3}$. Note that $\angle BEF = 90^\circ$ and $BE = 4\sqrt{2}$ so the area of $\triangle BEF$ (the shaded triangle) is

$$\frac{1}{2}BE \cdot EF = \frac{1}{2} \cdot 4\sqrt{2} \cdot \frac{4\sqrt{2}}{3} = \frac{16}{3}.$$

Answer to F6: $16/3$

F7

Define $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. (This is the Fibonacci sequence, and it starts $1, 1, 2, 3, 5, 8, \ldots$.) Find $\gcd(F_{30}, 30)$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b. You may use the fact that $30 = 2 \times 3 \times 5$.

Solution. It suffices to check whether each of 2, 3, and 5 divides F_{30}. Using the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, we compute some initial values for F_n modulo 2, 3, and 5.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$F_n \bmod 2$	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1	0
$F_n \bmod 3$	1	1	2	0	2	2	1	0	1	1	2	0	2	2	1	0	1	1	2	0	2
$F_n \bmod 5$	1	1	2	3	0	3	3	1	4	0	4	4	3	2	0	2	2	4	1	0	1

From this we see that $\{F_n \bmod 2\}_n$ repeats with period 3, $\{F_n \bmod 3\}_n$ repeats with period 8, and $\{F_n \bmod 5\}_n$ repeats with period 20. Hence, $F_{30} \equiv F_3 \equiv 0 \pmod 2$, $F_{30} \equiv F_6 \equiv 2 \pmod 3$, and $F_{30} \equiv F_{10} \equiv 0 \pmod 5$, so $\gcd(F_{30}, 30) = 2 \cdot 5 = 10$.

Answer to F7: 10

F8

For all positive integers n, let $f(n)$ be a integer whose value depends on n. Suppose that $f(1) = 1$ and that $f(2n) = 2f(n) - 1$ and $f(2n + 1) = 2f(n) + 1$ for all n. Compute $f(1023)$.

Solution. To find $f(1023)$, we repeatedly apply the rules we're given until we reach the base case $f(1) = 1$. It turns that for $f(1023)$, we only ever need to apply the rule $f(2n+1) = 2f(n) + 1$.

$$\begin{aligned}
f(1023) &= f(2 \cdot 511 + 1) \\
&= 2f(511) + 1 \\
&= 2f(2 \cdot 255 + 1) + 1 \\
&= 4f(255) + 3 \\
&= 8f(127) + 7 \\
&= 16f(63) + 15 \\
&= 32f(31) + 31 \\
&= 64f(15) + 63 \\
&= 128f(7) + 127 \\
&= 256f(3) + 255 \\
&= 512f(1) + 511 \\
&= 512 + 511 \\
&= 1023.
\end{aligned}$$

Answer to F8: 1023